An Improved Outer Bound for Multisource Multisink Network Coding

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***Abstract*— The Max-flow Min-cut bound is a fundamental result in the theory of communication networks, which characterizes the optimal throughput for a point-to-point communication network. The recent work of Ahlswede *et al* [1] extended it to the single-source multisink multicast networks and Li *et al* [2] proved that it can be achieved by linear codes. Following this line, Erez, Feder [3] and Ngai, Yeung [4] both proved that the Max-flow Min-cut bound remains tight in the single-source two-sink non-multicast networks. But the Max-flow Min-cut bound is in general quite loose [5]. In this work we prove an improved outer bound, named network sharing bound, for a special class of networks, the three-layer networks. We further show that the network sharing bound implies that the coding among messages from different sources has no benefit if the goal is to minimize the total needed bandwidth.**

I. Introduction

A directed network is defined as $G = (V, E)$, where V is the set of nodes and E is the set of directed edges. A three-layer network is a special network G of which all the nodes are lined up in a three-layer architecture. Its prototype was first formulated in [6] as a distributed source coding system, which consists of multiple sources, multiple encoders and multiple decoders. Each encoder has access to a certain subset of the sources, each decoder has access to a certain subset of the encoders, and each decoder reconstructs a ceratin subset of the sources. A three-layer network is a distributed source coding system in the network format. An example of a distributed source coding system is shown in Fig. 1, where its equivalent three-layer network is shown in Fig. 2. In a three-layer network, each directed edge is assumed to be error free and thus is called an error-free channel. All channels except the coding channels $(e.g., (1, 1'), (2, 2'), (3, 3')$ in Fig. 2), as will be defined later, are considered straight connections, therefore there is no constraint on the capacities of these channels. However, the information flow on each coding channel is limited by its capacity. The source nodes transmit independent messages to the sink nodes under the channel capacity constraints to meet the sink demands. We are interested in characterizing the achievable information rate region at the sources for which the demands at the sinks are satisfiable.

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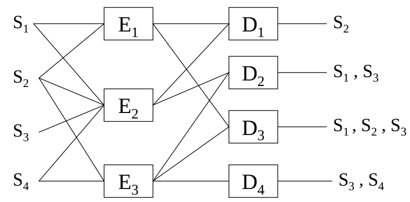

Fig. 1. An Example of Distributed Source Coding System

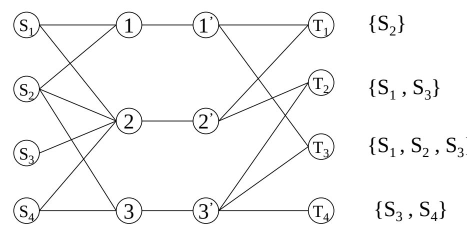

Fig. 2. An Example of Three-layer Network

For the single-source multisink multicast networks, Ahlswede *et al* showed that the source information rate region can be characterized by the Max-flow Min-cut bound. As a first step toward the multisource multisink network, Erez, Feder and Ngai, Yeung recently proved that the Max-flow Min-cut bound remains tight in the single-source two-sink non-multicast networks. Although it is obvious that the Max-flow Min-cut bound serves as an outer bound for the achievable rate region, it is in general not tight in arbitrary multisource multisink networks.

The main result of the paper is an improved outer bound over the Max-flow Min-cut bound for the achievable rate region of three-layer networks. This new bound is originally found by analyzing the role of the so-called side information at the decoders. Suppose that the information sources in the three-layer network are $X_1, X_2, \ldots, X_k$. If the channel output of a coding channel is a function of the source data in a subset $\{X_i : i \in \alpha\}$ and it is available for a sink which is required to decode the source data in a subset $\{X_j : j \in \gamma\}$, then the output of this channel is said to be a side information for the decoder if $\alpha \cap \gamma = \phi$. The following example explains the role of side information at the decoder.

We consider the classical example of network coding G in Fig. 3 which consists of two sources and two sinks. Each edge in G has unit capacity. Messages X and Y are generated at source node S_1 and S_2 respectively. Each of X and Y is of one bit. In order to decode Y at sink node T_1 and X at sink node T_2, without loss of generality, suppose that $X+Y$ is sent on edge (c_2, r_2) and X is sent on edge (c_1, r_1). We observe that Y can be decoded at T_1 with the help of side information X, while X can not be decoded at T_2 for missing the side information Y.

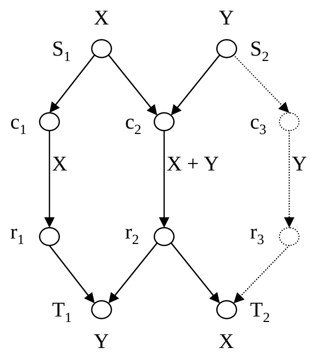

Fig. 3. Two-source Two-sink Network with Partial Side Information

Now, we take a further look at the information rate region of this network. It is easy to see that the Max-flow Min-cut bound of the given network is

$$\begin{aligned} R_X &\leq C(c_2, r_2) = 1 \\ R_Y &\leq C(c_2, r_2) = 1 \\ R_X + R_Y &\leq C(c_1, r_1) + C(c_2, r_2) = 2, \end{aligned}$$

which is not achievable from the previous analysis. The fact of missing enough side information at sink node T_2 suggests that a tighter outer bound may be obtained by analyzing the role of side information. Actually we have

$$\begin{aligned} R_X &\leq C(c_2, r_2) = 1 \\ R_Y &\leq C(c_2, r_2) = 1 \\ R_X + R_Y &\leq C(c_2, r_2) + \min\{C(c_1, r_1), C(c_3, r_3)\} \\ &= 1 + \min\{1, 0\} = 1. \end{aligned}$$

Obviously, this bound is tight. If we add one more channel to provide Y at sink T_2 as in Fig. 4, then X and Y can be both decoded at T_2 and T_1 respectively. Hence, the availability of enough side information at decoders is essential for the decodability.

The rest of the paper is organized as follows: In Section II, we give a formal problem formulation and introduce the notions used throughout the paper. In Section III, we first give the network sharing bound for the three-layer network with one-to-one source-sink transmission, then extend it to arbitrary three-layer network. The corresponding proofs are given in Section IV. In section V, we show by examples that network sharing bound is still not tight in general three-layer network. Conclusions are given in Section VI.

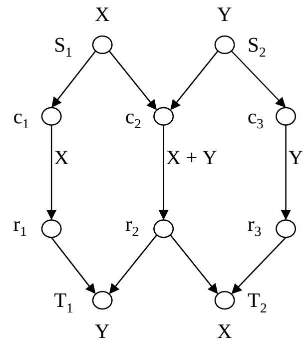

Fig. 4. Two-source Two-sink Network with Full Side Information

II. Network Model

Let us now present our network model. A three-layer network consists of the following elements:

1) S, the index set of source nodes; the source nodes are denoted by s_i with $i \in S$;
2) T, the index set of sink nodes; the sink nodes are denoted by t_i with $i \in T$;
3) I, the index set of the coding channels;
4) $P = \{c_i : i \in I\}$, the set of coding nodes;
5) $R = \{r_i : i \in I\}$, the set of relay nodes;
6) $A = \{A_i : A_i \subseteq S, i \in I\}$, the set of connections between the source nodes and the coding nodes;
7) $B = \{B_i : B_i \subseteq T, i \in I\}$, the set of connections between the relay nodes and the sink nodes;
8) $E = \{e_i, i \in I\}$, the set of the coding channels;
9) $C = \{C_i, i \in I\}$, the set of capacities of the coding channels;
10) $D_i \in 2^S \setminus \phi, i \in T$, which specify the reconstruction requirements of the sink nodes t_i.

The jth source which is available at the source node s_j is denoted by $\mathbf{X}_j = \{X_{jk}\}_{k=1}^{\infty}$. We assume that $\mathbf{X}_j, j = 1, \cdots, N$ are independent, and $X_{jk}, k = 1, 2, \ldots$ are independent and identically distributed (i.i.d.) copies of a generic random variable X_j with alphabet $\mathcal{X}_j$, where $|\mathcal{X}_j| < \infty$. We assume no capacity constraint on the channels other than the coding channels. The sets $A_i, B_i, i = 1, \cdots, |I|$ and $D_j, j = 1, \cdots, |T|$ specify the three-layer network as follows: A coding node c_i has access to s_j if and only if $j \in A_i$, the sink node t_j has access to a relay node r_i if and only if $j \in B_i$, and t_j reconstructs $X_l, l \in D_j$.

In the three-layer network model, we may assume that $i \neq j$ implies $(A_i, B_i) \neq (A_j, B_j)$. Otherwise, we may merge the coding channels i and j to a new channel with capacity $C_i + C_j$. Based on this observation, we may index the coding channels, the capacities, even the coding nodes and the relay nodes by the sets A_i and B_j. That is, we may use the notations $e(\alpha, \beta), C(\alpha, \beta)$, even $c(\alpha, \beta)$ and $r(\alpha, \beta)$ in place of e_i, C_i, c_i and r_i, when $A_i = \alpha, B_j = \beta$. Furthermore, we may assume for all $(\alpha, \beta), \alpha \neq \phi, \beta \neq \phi$, there exists a coding channel $e(\alpha, \beta)$. The absence of such a channel can be specified by letting $C(\alpha, \beta) = 0$. Therefore, we may assume that I consists all possible (α, β) where α is a non-empty sub-set of the index set S and β is a non-empty subset of T.

In our model, A_i contains the indices of source nodes that are accessible by c_i, B_j contains the sink nodes which are accessible by relay node r_j. Let

$$d_i : \left(\prod_{j \in D_i} \mathcal{X}_j\right) \times \left(\prod_{j \in D_i} \mathcal{X}_j\right) \to \{0, 1\}$$

be the Hamming distortion measure $i \in T$; *i.e.*, for any $\mathbf{x}$ and $\mathbf{x}'$ in $(\prod_{j \in D_i} \mathcal{X}_j) \times (\prod_{j \in D_i} \mathcal{X}_j)$

$$d_i(\mathbf{x}, \mathbf{x}') = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{x}', \\ 1, & \text{if } \mathbf{x} \neq \mathbf{x}'. \end{cases}$$

Let $X_j^n = (X_{j1}, \ldots, X_{jn}), I_m = \{i : i \in I, m \in B_i\}$. An $(n, (\eta_l, l \in I), (\triangle_i, i \in T))$ code is defined by

$$F_l : \prod_{j \in A_l} \mathcal{X}_j^n \to \{0, 1, \ldots, \eta_l - 1\}, \quad l \in I$$

$$G_m : \prod_{l \in I_m} \{0, 1, \ldots, \eta_l - 1\} \to \prod_{j \in D_m} \mathcal{X}_j^n, \quad m \in T$$

and

$$\triangle_i = n^{-1} E \sum_{k=1}^{n} d_i((X_{jk}, j \in D_i), (\hat{X}_{jk}, j \in D_i)), i \in T$$

where

$$(\hat{X}_j^n, j \in D_m) = G_m(F_l(X_j^n, j \in A_l), l \in I_m).$$

An $|I|-$tuple $(R_l, l \in I)$ is admissible if for every $\epsilon > 0$, there exists for sufficiently large n an $(n, (\eta_l, l \in I), (\Delta_i, i \in T))$ code such that

$$n^{-1} \log \eta_l \leq R_l, \text{ for all } l \in I$$

and

$$\Delta_i \leq \epsilon, \text{ for all } i \in T.$$

Let $\boldsymbol{R} = (R_l, l \in I)$, and let

$$\mathcal{R} = \{\boldsymbol{R} : \boldsymbol{R} \text{ is admissible}\}$$

be the admissible coding rate region. If the capacity vector $\boldsymbol{C} \in \mathcal{R}$, then we say that the transmission problem of the source over the network with the given demand at sinks is resolvable.

The goal of this paper is to characterize $\mathcal{R}$. In the next section, we give an outer bound for $\mathcal{R}$. It needs to be pointed out that while 1)the outer bound $\mathcal{R}_{out}$ given in [6] can not be evaluated explicitly, 2) although the bound $\mathcal{R}_{LP}$ in [6] can be evaluated, its evaluation is involved, our outer bound is much more explicit and much easier to understand.

III. Main Results

In this section, we first prove our bound for three-layer networks with one-to-one source-sink transmission. Then we extend it to arbitrary three-layer networks.

Definition 1: Suppose $\gamma \subseteq \{1, 2, ..., |S|\}, \gamma \neq \phi$ and an order in γ denoted by $\{i_1 \prec i_2 \prec \cdots \prec i_m\}, m \leq |S|$, we define $i_\beta \triangleq \min\{\beta \cap \gamma\}$ and $\gamma_\beta = \{i \in \gamma : i \prec i_\beta\}, \beta \subseteq T, \beta \neq \phi$.

Theorem 1: Given a three-layer network with one-to-one source-sink transmission (i.e. $T = S$ and $D_i = \{i\}, i \in T$), if the transmission problem is resolvable, then for any nonempty subset $\gamma \subseteq \{1, 2, \ldots, |S|\}$ and any order $\prec$ in γ,

$$\sum_{i \in \gamma} H(\mathcal{X}_i) \leq \sum_{\substack{(\alpha,\beta): \beta \cap \gamma \neq \phi, \\ \alpha \cap \gamma \not\subseteq \gamma_\beta}} C(\alpha, \beta).$$

We call this bound the network-sharing bound. For any arbitrary three-layer network with one-to-one source-sink transmission, the network sharing bound is an improvement over the Max-flow Min-cut bound, i.e.,

$$\sum_{\substack{(\alpha,\beta): \beta \cap \gamma \neq \phi, \\ \alpha \cap \gamma \not\subseteq \gamma_\beta}} C(\alpha, \beta) \leq \sum_{\substack{(\alpha,\beta): \beta \cap \gamma \neq \phi, \\ \alpha \cap \gamma \neq \phi}} C(\alpha, \beta).$$

From the proof of our theorem, we believe that this bound is implied by the linear programming bound in [6].

In a general three-layer network, each sink can request messages from multiple sources. In such networks, our converse proof with one-to-one source-sink transmission in IV seems inapplicable. However, this problem can be easily solved by sink decomposition, *i.e.*, we decompose each sink t_i into $|D_i|$ copies, each of them has a single source reconstruction demand and has the same set of connections with the coding channels as t_i. For example, by sink decomposition, the network in Fig. 2 can be transformed into the network in Fig. 5. Thus, any general three-layer network with arbitrary sink demands can be viewed as a three-layer network with one-to-many source-sink transmission. Therefore, by enumerating all possible $|S|$ tuples $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{|S|})$ where $\mathcal{T}_i$ is a set of sink nodes at which the ith source is to be decoded, we get $\prod_{j \in \{1, \ldots, |S|\}} |T_j|$ one-to-one source-sink transmission subnetworks. For any coding scheme such that the general three-layer network is decodable, all the corresponding one-to-one source-sink transmission subnetworks are decodable. Therefore, the minimum of the network sharing bounds of these subnetworks gives the network sharing bound of the general three-layer network.

Definition 2: For any three-layer network G with one-to-many source-sink transmission, $\forall (j_1, j_2, \ldots, j_{|S|}), j_i \in \mathcal{T}_i$, we define $G_{(j_1, j_2, \ldots, j_{|S|})}$ as a three-layer subnetwork of G with one-to-one source-sink transmission.

Corollary 1: Given an arbitrary three-layer network, if the source transmission problem is resolvable, then for any sub-network G with one-to-one source-sink transmission, by re-indexing the channels with the connections in the sub-network, for any nonempty subset $\gamma \subseteq \{1, 2, \ldots, |S|\}$,

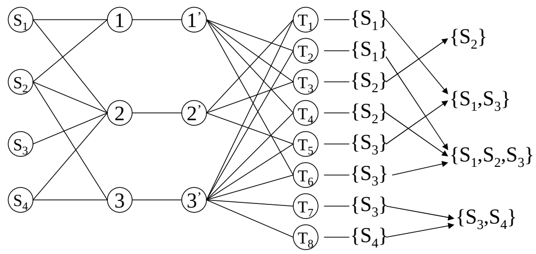

Fig. 5. Sink Decomposition for Arbitrary Three-layer Network

$$\sum_{i\in\gamma} H(\mathcal{X}_i) \le \min_{G,\prec} \sum_{(\alpha,\beta):\, \substack{\beta\cap\gamma\neq\phi,\\ \alpha\cap\gamma\not\subseteq\gamma_\beta}} C(\alpha,\beta),$$

where $\prec$ is the order in γ.

We also note that a recent result [7] has extended the network sharing bound to any arbitrary multisouce multisink network.

An important consequence of the network sharing bound is the following observation. The network sharing bound implies that

$$\inf\left\{\sum_{i\in I} R_i : \boldsymbol{R}\in\mathcal{R}\right\} = \sum_{j\in S} H(X_j).$$

This result means that, at least in the one-to-one source-sink transmission case, coding among messages from different sources has no benefit at all, if our goal is to minimize the total data rate of all channels in the network. This point can be seen from the following intuition: suppose that we code messages from different sources, then the data rate for some of the side-information channels (channels satisfying $\alpha\cap\beta=\phi$) must be non-zero. This is a very important observation since it implies that single source multicast might be sufficient to achieve minimum total transmission cost.

Let the total rate of the side-information channels be

$$R_s \triangleq \sum_{\alpha\cap\beta=\phi} R(\alpha,\beta),$$

then we have

Corollary 2:

$$\sum_{i\in S} H(\mathcal{X}_i) \le \sum_{(\alpha,\beta)\in I} R(\alpha,\beta) - 2^{-|S|} R_s.$$

Proof: Let $\gamma = S$ in Theorem 1, we can obtain

$$\begin{aligned}
&\sum_{i\in S} H(\mathcal{X}_i)\\
\overset{(1)}{\le}\ & \sum_{(\alpha,\beta):\,\alpha\cap\beta\neq\phi} R(\alpha,\beta) + \min_{\prec} \sum_{(\alpha,\beta):\,\substack{\alpha\cap\beta=\phi,\\ \alpha\not\subseteq\gamma_\beta}} R(\alpha,\beta)\\
\le\ & \sum_{(\alpha,\beta):\,\alpha\cap\beta\neq\phi} R(\alpha,\beta) + \frac{1}{|S|!}\sum_{\prec} \sum_{(\alpha,\beta):\,\substack{\alpha\cap\beta=\phi,\\ \alpha\not\subseteq\gamma_\beta}} R(\alpha,\beta)\\
\overset{(2)}{=}\ & \sum_{(\alpha,\beta):\alpha\cap\beta\neq\phi} R(\alpha,\beta)\\
& +\frac{1}{|S|!} \sum_{(\alpha,\beta):\,\alpha\cap\beta=\phi} |S|!\left(1-\frac{1}{\binom{|\alpha|+|\beta|}{|\alpha|}}\right) R(\alpha,\beta)\\
\overset{(3)}{\le}\ & \sum_{(\alpha,\beta):\alpha\cap\beta\neq\phi} R(\alpha,\beta) + \sum_{(\alpha,\beta):\,\alpha\cap\beta=\phi} (1-2^{-|S|}) R(\alpha,\beta)\\
=\ & \sum_{(\alpha,\beta)\in I} R(\alpha,\beta) - 2^{-|S|} R_s,
\end{aligned}$$

where step (1) follows from the fact that $\alpha\cap\beta\neq\phi$ implies $\alpha\not\subseteq\gamma_\beta$. (2) follows from that for all α,β satisfying $|\alpha|=a, |\beta|=b, \alpha\cap\beta=\phi$, the total number of (α,β) is $\binom{|S|}{a,b,|S|-a-b}$; furthermore, for a fixed order $\prec$ in S, the number of (α,β) satisfying $|\alpha|=a, |\beta|=b, \alpha\cap\beta=\phi$ and $\alpha\subseteq\gamma_\beta$ is $\binom{|S|}{a+b}$. This is obtained when we choose α and β jointly with the fixed order subject to $\alpha\subseteq\gamma_\beta$. From the property of symmetry, a portion of

$$\frac{\binom{|S|}{a+b}}{\binom{|S|}{a,b,|S|-a-b}} = \frac{1}{\binom{|\alpha|+|\beta|}{|\alpha|}}$$

should be excluded from the bound, and Step (3) follows from the fact

$$\binom{|\alpha|+|\beta|}{|\alpha|} \le 2^{|\alpha|+|\beta|} \le 2^{|S|}.$$

IV. Proof of Main Result

Proof of Theorem 1: Suppose the $|I|$-tuple $\boldsymbol{C}$ is admissible for the given three-layer network, for every $\epsilon>0$, there exists for sufficiently large n an $(n, (\eta_l, l\in P), (\Delta_i, i\in T))$ code such that

$$n^{-1}\log\eta_l \le C_l,\ \text{for all } l\in I$$

and

$$\Delta_i \le \epsilon,\ \text{for all } i\in T.$$

Actually every $C_l, l\in I$ corresponds to a $C(\alpha,\beta)$ where $\alpha = A_l$ and $\beta = B_l$. Define $U(\alpha,\beta) = F_l(X_j^n, j\in\alpha)$, we have

1) $H(U(\alpha,\beta)) \le \log\eta_l \le nC(\alpha,\beta)$;
2) $H(U(\alpha,\beta)|X_j^n : j\in\alpha) = 0$;
3) By Fano's inequality, $\exists\delta$ depending on ϵ such that $\delta\to 0$ as $\epsilon\to 0$ and for any $i\in\beta$, $n^{-1}H(X_i^n|\{U(\alpha,\beta) : i\in\beta,\ \text{all } \alpha\neq\phi\}) \le \delta$.

We have

$$\begin{aligned}
& \sum_{(\alpha,\beta):\,\substack{\beta\cap\gamma\neq\phi,\\ \alpha\cap\gamma\not\subseteq\gamma_\beta}} nC(\alpha,\beta) + \sum_{i\notin\gamma} H(X_i^n)\\
\ge\ & \sum_{(\alpha,\beta):\,\substack{\beta\cap\gamma\neq\phi,\\ \alpha\cap\gamma\not\subseteq\gamma_\beta}} H(U(\alpha,\beta)) + \sum_{i\notin\gamma} H(X_i^n)\\
\ge\ & H(\{U(\alpha,\beta) : \beta\cap\gamma\neq\phi, \alpha\cap\gamma\not\subseteq\gamma_\beta\}, \{X_i^n : i\notin\gamma\})\\
=\ & H(\{U(\alpha,\beta) : \beta\cap\gamma\neq\phi, \alpha\cap\gamma\not\subseteq\gamma_\beta\}, \{X_i^n : i\in S\})\\
& -H(\{X_j^n : j\in\gamma\}|U(\alpha,\beta) : \beta\cap\gamma\neq\phi, \alpha\cap\gamma\not\subseteq\gamma_\beta\},\\
& \qquad \{X_i^n : i\notin\gamma\})\\
\overset{(a)}{\ge}\ & H(\{X_i^n : i\in S\}) - n|\gamma|\delta\\
=\ & \sum_{i\in\gamma} H(X_i^n) + \sum_{i\notin\gamma} H(X_i^n) - n|\gamma|\delta.
\end{aligned}$$

Therefore,

$$\sum_{i\in\gamma} n^{-1}H(X_i^n) \le \sum_{\substack{(\alpha,\beta):\ \beta\cap\gamma\neq\phi,\\ \alpha\cap\gamma\nsubseteq\gamma_\beta}} C(\alpha,\beta) + |\gamma|\delta.$$

Let $\epsilon \to 0$ and $n \to \infty$, then $\delta \to 0$ and we complete the proof, where the key step (a) is proved as follows:
Let $\gamma = \{i_1 \prec i_2 \prec \cdots \prec i_{|\gamma|}\}$, for convenience of notation, we define a set $\Delta = \{U(\alpha,\beta) : \beta\cap\gamma \neq \phi, \alpha\cap\gamma \nsubseteq \gamma_\beta\}$ and another set $\Lambda = \{X_i^n : i \notin \gamma\}$. Thus from the chain rule of entropy functions

$$H(\{X_j^n : j\in\gamma\}|\Delta,\Lambda) = \sum_{k=1}^{|\gamma|} H(X_{i_k}^n|X_{i_1}^n,...,X_{i_{k-1}}^n,\Delta,\Lambda),$$

so we only need to show that

$$\begin{aligned}
&H(X_{i_k}^n|X_{i_1}^n,...,X_{i_{k-1}}^n,\Delta,\Lambda)\\
&\le H(X_{i_k}^n|X_{i_1}^n,...,X_{i_{k-1}}^n,\{U(\alpha,\beta) : i_k\in\beta,\\
&\qquad \alpha\cap\gamma \nsubseteq \gamma_\beta\},\Lambda)\\
&\overset{(b)}{=} H(X_{i_k}^n|X_{i_1}^n,...,X_{i_{k-1}}^n,\Lambda,\bigcup_{t=1}^{k}\{U(\alpha,\beta) :\\
&\qquad i_\beta = i_t, i_k\in\beta, \alpha\cap\gamma \nsubseteq \{i_1,...,i_{t-1}\}\})\\
&\overset{(c)}{\le} H(X_{i_k}^n|\bigcup_{t=1}^{k}\{U(\alpha,\beta) : i_\beta = i_t, i_k\in\beta,\\
&\qquad \alpha\cap\gamma \subseteq \{i_1,...,i_{t-1}\}\},\bigcup_{t=1}^{k}\{U(\alpha,\beta) :\\
&\qquad i_\beta = i_t, i_k\in\beta, \alpha\cap\gamma \nsubseteq \{i_1,...,i_{t-1}\}\})\\
&= H(X_{i_k}^n|\bigcup_{t=1}^{k}\{U(\alpha,\beta) : i_\beta = i_t, i_k\in\beta\})\\
&\overset{(d)}{=} H(X_{i_k}^n|U(\alpha,\beta) : i_k\in\beta)\\
&\le n\delta.
\end{aligned}$$

The noted steps are explained as follows:

(b) This union includes all the β, such that $i_k \in \beta$.

(c) By the fact 2), it follows that $H(U(\alpha,\beta) : \alpha\cap\gamma \subseteq \{i_1,...,i_{k-1}\}|X_{i_1}^n,...,X_{i_{k-1}}^n,\Lambda) = 0, \forall k \in \{1,...,|\gamma|\}, \forall\beta \subseteq \{1,...,|T|\}, \beta\neq\phi$. and $H(Y|X) \le H(Y|g(X))$.

(d) Similar as (b).

Therefore, Theorem 1 is proved.

V. Examples

It is easy to show that the network sharing bound is tight in the case of one-to-one source-sink transmission with two sources and two sinks. However, the following example shows that the bound is no longer tight for three sources.

Example 1: In this example, we show the significant improvement of network sharing bound over the Max-flow Min-cut bound as an outer bound in some special cases. However, this example also proves that the network sharing bound is not tight for the general three-layer network.

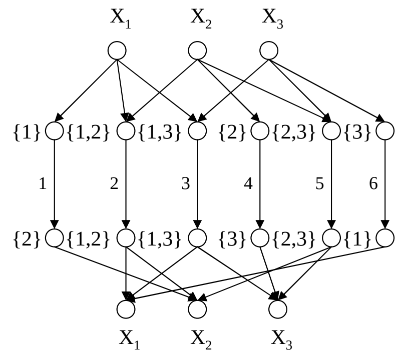

Fig. 6. An Example of Three-source Three-sink Three-layer Network

Consider the three-layer network G in Fig. 6, where

$$\begin{aligned}
&S = T = \{1,2,3\},\\
&I = \{1,2,3,4,5,6\}, I' = \{1',2',3',4',5',6'\},\\
&A_1 = \{1\}, A_2 = \{1,2\}, A_3 = \{1,3\},\\
&A_4 = \{2\}, A_5 = \{2,3\}, A_6 = \{3\},\\
&B'_1 = \{2\}, B'_2 = \{1,2\}, B'_3 = \{1,3\},\\
&B'_4 = \{3\}, B'_5 = \{2,3\}, B'_6 = \{1\},\\
&D_1 = \{1\}, D_2 = \{2\}, D_3 = \{3\},\\
&C(\{1,2\},\{1,2\}) = 1, C(\{1,3\},\{1,3\}) = 1,\\
&C(\{2,3\},\{2,3\}) = 1, C(\{1\},\{2\}) = 1,\\
&C(\{2\},\{3\}) = 1, C(\{3\},\{1\}) = 1.
\end{aligned}$$

The Max-flow Min-cut bound can be easily obtained as follows

$$\begin{aligned}
&H(X_1) \le 2\\
&H(X_2) \le 2\\
&H(X_1) + H(X_2) \le 4\\
&H(X_1) + H(X_3) \le 4\\
&H(X_2) + H(X_3) \le 4\\
&H(X_1) + H(X_2) + H(X_3) \le 6.
\end{aligned}$$

Now, lets examine the network sharing bound. Since the given network is symmetric, we can just examine one order, without loss of generality, $\gamma = \{1 \prec 2 \prec 3\}$. Since the bounds of subsets of two or fewer source nodes are easy to be checked, here we just give the derivation for the bound with three sources. We enumerate all the β sets as follows

$$\begin{aligned}
&\beta = \{1\}, \gamma_\beta = \phi, \forall\alpha\subseteq\gamma \to C(\{3\},\{1\}) = 1;\\
&\beta = \{1,2\}, \gamma_\beta = \phi, \forall\alpha\subseteq\gamma \to C(\{1,2\},\{1,2\}) = 1;\\
&\beta = \{1,3\}, \gamma_\beta = \phi, \forall\alpha \to C(\{1,3\},\{1,3\}) = 1;\\
&\beta = \{2\}, \gamma_\beta = \{1\}, \alpha\cap\gamma \nsubseteq \{1\} \not\leftrightarrow \alpha\cap\gamma = \{1\};\\
&\beta = \{2,3\}, \gamma_\beta = \{1\}, \alpha\cap\gamma \nsubseteq \{1\} \to C(\{2,3\},\{2,3\}) = 1;\\
&\beta = \{3\}, \gamma_\beta = \{1,2\}, \alpha\cap\gamma \nsubseteq \{2,3\} \not\leftrightarrow \alpha\cap\gamma = \{2\},
\end{aligned}$$

where $\not\leftrightarrow$ means "contradicts". Thus the network sharing bound is

$$H(X_1) \le 2$$

$$H(X_2) \leq 2$$
$$H(X_1) + H(X_2) \leq 3$$
$$H(X_1) + H(X_3) \leq 3$$
$$H(X_2) + H(X_3) \leq 3$$
$$H(X_1) + H(X_2) + H(X_3) \leq 4,$$

which suggests a significant improvement over the Max-flow Min-cut bound. However, it is not hard to see that the information rate triple $(2, 1, 1)$ is not achievable in any order of γ . In fact, the last inequality should be replaced by

$$H(X_1) + H(X_2) + H(X_3) \leq 3$$

to make the bound tight.

Therefore, despite of the potential significant improvement the network sharing bound could offer over the Max-flow Min-cut bound, it is still not tight in the general three-layer network.

VI. Conclusion

In this paper, we proved an improved outer bound of the admissible rate region for a special class of multisouce multisink network, namely, the three-layer network by analyzing the role of side information. Although the proposed network sharing bound is not tight for the general three layer network, it provides significant improvement over the Max-flow Min-cut bound. Another important consequence is that the network sharing bound implies that network coding among messages from different sources has no benefit if our goal is to minimize the total bandwidth needed. Based on this result, we conjecture that under reasonable assumptions this conclusion holds for arbitrary multi-source, multi-sink networks.

References

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1204–1216, July 2000.

[2] S.-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory*, vol. 49, pp. 371–381, Feb. 2003.

[3] E. Erez and M. Feder, "Capacity region and network codes for two receivers multicast with private and common data," *Workshop on Coding, Cryptography and Combinatorics*, 2003.

[4] C. K. Ngai and R. W. Yeung, "Multisource network coding with two sinks," *International Conference on Communications, Circuits and Systems (ICCCAS)*, June 2004.

[5] R. W. Yeung, *A First Course in Information Theory*. New York: Kluwer/Plenum, 2002.

[6] R. W. Yeung and Z. Zhang, "Distributed source coding for satellite communications," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1111–1120, May 1999.

[7] X. Yan, J. Yang, and Z. Zhang, "An outer bound for multisource multisink network coding and its relation to minimum cost network coding," *to be submitted to IEEE Trans. Inform. Theory*, 2005.

[8] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[9] R. W. Yeung and Z. Zhang, "On symmetrical multilevel diversity coding," *IEEE Trans. Inform. Theory*, vol. 45, pp. 609–621, Mar. 1999.

[10] R. W. Yeung and C. K. Ngai, "Two approaches to quantifying the bandwidth advantage of network coding," *presented at 2004 IEEE Information Theory Workshop*, Oct. 2004.